Recursive Definitions

1. Generations

We often define more complex objects in terms of simpler objects. We can define more complex sets in terms of less complex sets. For example, consider Bob. We want to define all the descendents of Bob. They are stratified into generations. The zeroth generation is just the set that contains Bob himself. Thus:

Generation-0 = {Bob}.

The next generation is all the children of Bob. This is

Generation-1 = { x | x is a child of Bob}.

We can go on to define next generation as the grandchildren of Bob. These are all the children of the children of Bob. Hence they are all the children of the people in the previous generation. Thus

Generation-2 = { x | x is a child of any person in Generation-1}.

We can go on to associate each number with a generation of descendents. So we are defining a function from the numbers to the generations. It's inconvenient to use the whole word Generation for each generation. So we just abbreviate it as G. Thus G is a function that associates every number n with the n-th generation. The n-th generation is G(n). We can now re-write the generations like this:

G(0) = {Bob};

G(1) = { x | x is a child of any person in G(0)};

G(2) = { x | x is a child of any person in G(1)};

. . .

G(n+1) = { x | x is a child of any person in G(n)}.

We've defined the various generations of descendents of Bob. But how do we define the totality of these descendents? We define the totality by forming the union of all the generations. We are thus forming the union of all the G(n) for all finite n. This union is the generation at infinity. It includes all finite generations. It is the *accumulation* of all these generations. So we use G(ω) to denote this generation:

$$G(\omega) = G(0) \cup G(1) \cup \; \ldots G(n) \cup G(n+1) \ldots$$

It's clearly impossible to define G(ω) by writing out the whole infinite sequence of unions of generations. We write G(ω) as

G(ω) = the union of all the G(*n*) such that *n* is in ω.

In symbols, this is

G(ω) = the union of { G(*n*) | *n* is in ω};

$$G(\omega) = \cup\{ G(n) \mid n \in \omega\}.$$

We can use the recursion scheme to define the generations:

1. *Initial Rule*. For the initial number 0, there is an initial generation G(0). If we are definining the descendents of some person *p*, G(0) = {*p*}.

2. *Successor Rule*. For every succesor number *n*+1, there exists a successor generation G(*n*+1). Formally, G(*n*+1) = { *x* | *x* is a child of a person in G(*n*)}.

3. *Limit Rule*. For the limit number ω, there is a limit generation G(ω). This is the accumulation of all the previous generations: $G(\omega) = \cup\{ G(n) \mid n \in \omega\}$.

2. Hamming Spaces

Given some object, and some operations that change that object, you can define a system of derived objects. The resulting system of objects is a kind of space organized by a similarity relation. For example, let the objects be English words and the operation be any edit that involves exactly one letter (either adding, subtracting, or altering a single letter). Table 1 illustrates part of the system that is derived from "life".

life	lime	slime	slim	skim	skit	skirt	
		time	timed	timid			
			tame	fame	flame		
	line	lone	alone	clone	cone	cane	can
	live	alive					
		dive	dice	ice			
				lice	lick	slick	stick
						kick	
						tick	trick
						lack	lark
							black
		five	fire	firs	first	fist	list
		give	gave				
		love	glove	grove	grave	gravel	gavel
			lose	lost	cost	cast	cat
					fast	fact	fat
					most	moist	
			move	more	mare	mars	marsh
	wife	wise	wish	wash	wasp	rasp	grasp
		wire	wore	worn	warn	warm	farm
							worm
					torn	ton	
				worse	worst		

Table 1. The system of edits starting with “life”.

We now define a series of generations of words derived from an initial seed word. For any word x, the set of words derived from x via a single edit is E(x). Thus

$$E(x) = \{ y \mid y \text{ is derived from } x \text{ via a single edit} \}.$$

We can use the recursion scheme to define the generations:

1. *Initial Rule*. For the initial number 0, there is an initial generation W(0). If the initial word is p, then the initial generation W(0) = {p}. In our example, the initial word is “life”, so W(0) = {“life”}.

2. *Successor Rule*. For every succesor number n+1, there exists a successor generation W(n+1). Formally, $W(n+1) = \cup\{ E(x) \mid x \in W(n) \}$.

3. *Limit Rule*. For the limit number ω, there is a limit generation $W(\omega)$. This is the accumulation of all the previous generations: $W(\omega) = \cup\{ W(n) \mid n \in \omega\}$.

Given a system of objects, and an operation that maps the system into itself, we can define the *distance* between any two objects *x* and *y* as the minimal number of operations it takes to change *x* into *y*. This is a notion of distance based on similarity – closer is more similar and farther is less similar. You can apply this notion of distance to possible universes: how many changes does it take to go from one universe to another?

3. Empirical Knowledge

An important analysis of empirical verifiability is provided by Salmon (1966). Salmon intends his analysis to be an improvement of the old positivistic analyses of empirical verifiability. According to Salmon:

> To say that a statement is verified is to say that it is supported by evidence. To say that a statement is verifiable is to say that it could be supported by evidence. The evidence (actual or possible) plays the role of a premise – more exactly, the statement of the evidence is a premise – and the verified or verifiable statement is the conclusion. Of course, in many important instances the conclusion is not a deductive consequence of the premise but is inductively supported by it. (Salmon, 1966: 463).

As a rough draft of a principle of verifiability, Salmon (1966: 464) says: "a statement which is neither analytic nor self-contradictory is empirically verifiable if and only if it is either an observation-statement or the conclusion of a correct inductive or deductive argument from verifiable premises." Since arguments can be iterated, the result is a rich hierarchy of empirically verifiable statements. We'll formalize Salmon's hierarchy.

We start with a small change in terminology. It seems odd to say that the statements in the hierarchy are empirically verifiable (as if they were going to be checked by someone). It seems better to say that they are *empirically grounded*.

The set *Observations* is a set of observation statements. This set is also known as the observation base or the data base. The set *Maths* is a set of statements taken to be analytically true. For example, Maths contains the axioms and definitions of some range of mathematical theories and Maths is closed under deduction. For any set of statements in Maths, all the entailments of that set are in Maths. We obtain the initial level S(0) by joining all the statements in Observations with those in Maths:

$$S(0) = \text{Observations} \cup \text{Maths}.$$

Given some set of statements X, the set Arg(X) is the set of all statements that can be derived from X by means of some correct inductive or deductive argument. We take inferences to the best explanation to be inductive arguments.

Given some level of statements S(n), we form the next level S(n+1) by including all the statements that can be derived from S(n) and by including S(n) itself. We thus obtain the definition for all successor levels:

$$S(n+1) = \mathrm{Arg}(S(n)) \cup S(n).$$

To form the entire universe of empirically grounded statements, we just take the union of all the levels. This is the limit level. It is defined as

$$S(\omega) = \cup\{S(n) \mid n \in \omega\}.$$

So a statement is *empirically grounded* (in some observations and mathematics) iff it is a member of S(ω). A *theory* is a set of statements. A theory is *empirically grounded* (in some observations and mathematics) iff it is a subset of S(ω).

References

Salmon, W. C. (1966) Verifiablity and logic. In M. L. Diamond & T. V. Litzenburg (Eds.) (1975) 456-479.

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