Royce's Model of the Absolute

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ABSTRACT: At the end of the 19th century, Royce uses the mathematical ideas of his day to describe the Absolute as a self-representative system. Working closely with Royce's texts, I will develop a model of the Absolute that is both more thoroughly formalized and that is stated in contemporary mathematical language. As I develop this more formal model, I will show how structures found within it are similar to structures widely discussed in current analytic metaphysics. The model contains structures found in the recent analytic metaphysics of modality; it contains Democritean worlds as defined by Quine; it contains Turing-computable sequences; it contains networks of interacting software objects as defined by Dennett. Much of the content of recent analytic metaphysics is already implicit in Royce's study of the Absolute. Far from being an obsolete system, of historical interest only, Royce's metaphysics is remarkably relevant today.

1. Introduction

At the end of the 19th century, Josiah Royce participated in what has come to be called the *great debate* (Royce, 1897; Armour, 2005).[1] The great debate concerned issues in metaphysical theology. And since metaphysics was primarily idealistic, it dealt considerably with the relations between the divine Self and lesser selves.

After the great debate, Royce developed his idealism in his Gifford Lectures (1898-1900). These were published as *The World and the Individual*. At the end of the first volume, Royce added a Supplementary Essay. The Essay, continuing themes from the great debate, developed an intriguing mathematical model of the relations between the divine Self and lesser selves. The second volume went on to elaborate Royce's idealism, with many references to the Essay.[2] After *The World and the Individual*, Royce developed a large body of work in logic and mathematics (Royce, 1905, 1951; Burch & Royce, 1987). That later work has been the subject of many articles (Hocking, 1956; Kuklick, 1971; Martin, 1976; Crouch, 2004; Pratt, 2007; Burch, 2010; Crouch, 2011). And yet, despite the considerable interest in Royce's logical work, and the importance of the work in the Essay for Royce's metaphysics, the model of the Essay has been mostly neglected.[3]

I have two goals. My first goal is to more thoroughly formalize the model of the Essay (hereafter, the *Model*) and to present that Model using current mathematical language. My second goal is to link the Model to structures studied in recent analytic metaphysics. First, I will link the Model to recent analytic metaphysics of modality (that is, to the theory of possible worlds). Second, I will link the Model to recent work on the construction of realistic physical worlds out of purely mathematical objects. These types of constructions are often associated with Quine and are closely associated with structures studied in current computational metaphysics. An important result is that any Turing-computable physical structure is found in the Model. This result goes a long way to showing that Roycean idealistic metaphysics is compatible with modern mathematical physics. Third, I will show how the Model supports networks of interacting software objects. The social networks discussed by Royce are found throughout the physical world. Finally, I will link the Model to recent analytic work in structuralism. All these links aim to serve one purpose: to show that Roycean metaphysics is not obsolete. On the contrary, one can do analytically exacting metaphysical work in a Roycean framework just as well as one can do such work within the frameworks of Quine, Plantinga, or David Lewis.

2. The Royce Map

I begin the discussion of the Model with the *perfectly accurate map of England within England*. It is clear that the definition is recursive: it repeats the structure of England within England. And it is equally clear that the perfect self-map of England is infinitely complex – it is an actual infinity of self-nested copies. Royce says:

> suppose that some one . . . assured us of this as a truth about existence, viz., "Upon and within the surface of England *there exists* somehow . . . an absolutely perfect map of the whole of England." . . . in this one assertion, "A part of England perfectly maps all England, on a smaller scale," there would be implied the assertion not now of a process of trying to draw maps, but of the contemporaneous presence, in England, of an infinite number of maps . . . The whole infinite series, possessing no last member, would be asserted as a fact of existence. . . . the perfect map of England, drawn within the limits of England, and upon a part of its surface, would, if really expressed, involve, in its necessary structure, the series of maps within maps such that no one of the maps was the last in the series. (SE 506-507)

For simplicity, suppose England is just a square crossed by a north-south road and an east-west road. (England ain't what it used to be.) Figure 1 shows the first four iterations of the Royce map. For Royce, the perfect map of England in England is an example of a *self-representative system* (SE 508-509). A self-representative system is one that is "precisely represented by a proper fraction or portion of itself" (SE 509). For instance, in Figure 1, the lower right quadrant of the map of England represents England itself. To formalize this idea, Royce uses the notion of a *one-to-one correspondence* between sets. Following Dedekind, he affirms that a self-representative system is a set that can be put into a one-to-one correspondence with one of its proper subsets (SE 510-512). Set S is a proper subset of set T iff every member of S is in T but not every member of T is in S.

As another illustration of a self-representative system, Royce presents the series of natural numbers: "the numbers form, in infinitely numerous ways, a self-representative system . . . the number-system . . . can be put into a one-to-one correspondence with one of its own constituent portions in any one of an endless number of ways." (SE 515) Let

N denote the set of natural numbers {0, 1, 2, 3, . . . }. Royce discusses several one-to-one functions from N into itself. He gives these examples: the function from *n* to 2*n*; the function from *n* to n^2; the function from *n* to the *n*-th prime (SE 515-517). Each of these is a self-map of N. For instance, the function that maps *n* to 2*n* puts the set of numbers into a one-to-one correspondence with one of its proper subsets, namely, the even numbers.

Each self-map *f* is a case in which a part of the number line is used to represent the entire number line. This is exactly like the case in which a part of England represents all of England.[4] Royce writes: "For just as, in the former case, the one purpose to draw the exact map of England within England, gave rise to the endless series of maps within maps, just so, I say, this one purpose involves of necessity the result that this second or representative series shall contain, as part of itself, an endless series of parts within parts" (SE 518-519). Royce again says the iteration of maps from N into itself "inevitably determines an endless [self-representative system] altogether parallel to our series of maps within maps of England. . . . Self-representation [of N by a map into itself] creates, at one stroke, an infinite chain of self-representations within self-representations" (SE 525).

From the examples of the perfect self-map of England and the self-maps of the natural numbers, Royce now generalizes. Following Dedekind, he tells us that a *Kette* is any structure (K, *f*) where K is a set and *f* is a function of K into K (SE 520). Although there are many types of Ketten, Royce says he is interested only in those in which *f* establishes a one-to-one correspondence between K and one of its proper subsets (SE 522-523).[5] And, among those Ketten, Royce will focus on those for which the set is the infinite set of natural numbers. Such Ketten have the form (N, *f*) where N is the natural numbers and *f* is a one-to-one function from N onto a proper subset of itself (SE 521-525).[6]

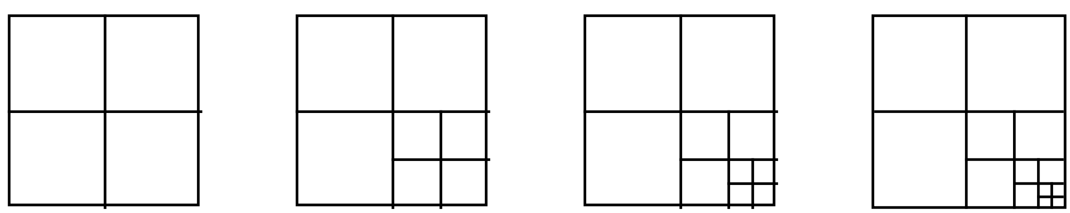

Figure 1. The first four iterations of a Roycean self-nested map.

3. The Absolute is a Self-Representative System

According to Royce's idealism, reality is ultimately a perfectly self-conscious mind. And Royce tells us he is mainly interested in "cases of self-representation, such as Self-consciousness" (SE 520). Any perfectly self-conscious mind is an ideal Self. As self-conscious, an ideal Self has some proper part of itself that perfectly represents itself; it has exactly the same infinitely self-nested structure as the exact self-map of England (SE 553). And, since any Kette (N, *f*) is also a self-representative system, any ideal Self is also analogous to such a Kette: "the number-series is a purely abstract image, a bare, dried skeleton, as it were, of the relational system that must characterize an ideally completed Self" (SE 526). Royce tells us explicitly that the structure of any Kette (N, *f*) is "precisely parallel to the structure of an ideal Self" (SE 535).[7] He writes

> The intellect has been studying itself, and, as the abstract and merely formal expression of the orderly aspect of its own ideally conceived complete Self, and of any ideal system that it is to view as its own deed, the intellect finds precisely the Number System, – not, indeed, primarily the cardinal numbers, but the ordinal numbers. Their formal order of first, second, and in general, of *next*, is an image of the life of sustained, or, in the last analysis, of complete Reflection. Therefore, this order is the natural expression of any recurrent process of thinking, and, above all, is due to the essential nature of the Self when viewed as a totality. (SE 538)

For Royce, the Absolute Self is an ideal Self, and as such "the Absolute must be a self-representative ordered system, or Kette" (SE 545). It has the form (N, *f*). He also says that the Absolute "*defines itself* as a self-representative system" (SE 545, my italics). Yet the self-consciousness of the Absolute is not restricted to any single type of self-representation. According to Royce, just as there can be many different perfect maps of England in England, or many self-maps of the number system into itself, so also there can be many different self-representations of the Absolute within the Absolute:

> As the England of our illustration could be self-mapped, if at all, then by countless series of various maps, not found in the same part of England and not in the least inconsistent with one another; and as the number-series – that abstract image of the bare form of every self-representative system of the type here in question, – can be self-represented in endlessly various ways, – so, too, the self-representation of the Absolute permitted by our view is confined to no one necessary case; but is capable of embodiment in as many and various cases of self-representation, in as many different forms of selfhood, each individual, as the nature of the absolute plan involves. So that our view of the Selfhood of the Absolute, if possible at all, leaves room for various forms of individuality within the one Absolute. (SE 546)

The Absolute represents itself to itself in many ways, each of which is a "form of selfhood" for the Absolute (SE 546). And these many forms of selfhood are different "forms of individuality within the one Absolute" (SE 546). On the Model that Royce is developing, the Absolute seems to *contain* all these forms of selfhood just as the set of all Ketten of the form (N, *f*) contains its members. Each form of selfhood in the Absolute is analogous to an infinite Kette (N, *f*). Obviously, these forms of selfhood are *not finite* (and they certainly are not *finite* human selves).[8] For, as Royce makes clear, "the word 'finite' is, with technical accuracy, used for systems that are *not* self-representative" (WI2 447, italics by Royce). On the contrary, as a self-representative Kette, each form of selfhood is an *infinite* form of selfhood; it is a form of absolute selfhood.

On the one hand, the Absolute contains infinitely many forms of absolute selfhood. On the other hand, one and only one of these forms is the Absolute Self. From out of the infinity of forms of absolute selfhood, Royce tells us that exactly one is selected to be the Absolute Self. This selection is self-selection: Royce says that the Absolute Self is a "a self-selected case of its type"(SE 566). This sounds like the Absolute Self paradoxically pulls itself up out of the Absolute by its own bootstraps. To avoid any paradox, it seems better to say that, among the infinitely many forms of absolute selfhood within the Absolute, exactly one is self-selective or self-affirmative, and that self-affirmative form is the Absolute Self.[9] The distinction between the Absolute Self and other forms of selfhood corresponds to the distinction between realized and unrealized (actualized and unactualized) forms of selfhood. Royce affirms that the Absolute Self is "an individual selection from an infinitely wealthy realm of unrealized possibilities" (WI2 448). The unselected forms remain unactualized as *barely possible* absolute selves (SE 566-569). The Absolute is *complete* with respect to these possible forms (CG 198, 208-209; SE 566-569, 580).

At this point, Royce has developed his mathematical *Model* of absolute selfhood. The Absolute contains every possible absolute self. Mathematically, every possible absolute self is a Kette of the type (N, *f*). Hence the total system of possible absolute selves is isomorphic to the set of all (N, *f*). Only one of these possible absolute selves is actual (it is the Absolute Self) while the others remain unactualized bare possibilities. To be sure, Royce does not literally identify the Absolute with the set of all Ketten of the type (N, *f*), nor does he literally identify the Absolute Self with one of these Ketten. On the contrary, he is clear that "the numbers, taken in abstract divorce from life, are mere forms" (SE 580). The Model merely outlines the skeleton of selfhood (SE 526, 527) and consists of only "dry bones" (SE 580). Nevertheless, Royce finds the Model useful, and refers to it from the start to the finish of the Second Volume of *The World and the Individual*. The mathematical form of the Absolute merits deep study. All formal work that follows is done strictly within the Model. However, since it is tedious to explicitly preface every sentence with "In the Model, . . .", this prefacing is now assumed where needed.

4. Thought-Streams in the Absolute Self

Royce tells us that a self contains a well-ordered discrete series of thoughts (WI2 105-107) and that every such series of thoughts is recursively generated (WI2 191-192). Throughout WI2, he refers to *recurrent processes* of thought. As a term of art, let us say these are *thought-streams*. Royce provides many examples of these thought-streams in the Essay. The series ⟨M, C(M), C(C(M)), . . . ⟩ is a thought-stream where M is any thought and C is any operation mapping thoughts onto thoughts (SE 496-497). Royce shows that Dedekind's *Gedankenwelt* determines a thought-stream (SE 511-513, 530). The thought-stream is ⟨S, TS, TTS, . . . ⟩ where S is one of my thoughts and T is the thought that S is one of my thoughts. Royce further illustrates this using the thought "Today is Tuesday" (SE 532-534). The series ⟨P, TP, TTP, . . . ⟩ where P is a thought and T is Bolzano's truth-operator forms is a thought stream (SE 544). The series ⟨P, KP, KKP, . . . ⟩ is a thought-stream where P is a thought and K is the knowledge operator (SE 578).

Within any Kette (N, f), Royce uses the numbers in N to model thoughts and the recursive application of f to model the passage from one thought to another. The series of prime numbers is like a thought-stream (SE 575-578). Royce defends the image of the absolute thought-process as "wandering from number to number" in a Kette (SE 587-588). Now, since every Kette contains N, the set of all Ketten of the form (N, f) can be thought of as the set N along with the set of all self-maps of N. If we let F be the set of all functions of N into itself, then each Kette is (N, f) such that f is a member of F. The set F models all the possible ways that the Absolute represents itself to itself. Hence there is some sense in which it is an ultimate totality of possibilities. Prior to the Essay, in *The Conception of God,* Royce said that the total system of possibilities is a *group* (CG 208-211). However, he did not elaborate. Here it is interesting to note that F forms a semigroup.[10]

Every Kette (N, f) determines a series of paths. For any Kette (N, f), every path through the Kette starts with some number n in N and recursively applies f. The path $s(f, n)$ in the Kette (N, f) is the series $\langle n, f(n), f(f(n)), \ . \ . \ . \rangle$. For example, suppose the Kette involves the doubling function d that maps n onto $2n$. Then the path $s(d, 1)$ is $\langle 1, 2, 4, 8, . . . \rangle$ while the path $s(d, 3)$ is $\langle 3, 6, 12, 24, . . . \rangle$. Within the Model, each path $s(f, n)$ in any possible absolute self is some possible absolute thought-stream. It is a possible absolute stream of consciousness or mental activity. Now let S be the set of all $s(f, n)$ such that f is in F and n is in N. Thus S is the set of all possible thought-streams in the Absolute. Some of these are within the Absolute Self (the actual Kette) while others lie in Ketten that remain unactualized as barely possible forms of absolute selfhood.

According to Royce, our world is ultimately one of the thought-streams in S (WI2 105-107). And, in his detailed discussion of possibility in the Essay, Royce argues that both our world and its counterfactual variants are also ultimately thought-streams in S (SE 567-569, 573-575, 580-581). Thus every possible ideal world is one of the thought-streams in S, so that S is the set of all possible ideal worlds. Of course, idealism asserts that every possible *physical* world is based on some ideal world; assuming this idealist hypothesis, the set of all possible worlds is identical to the set of all possible ideal worlds. Hence the set S is the set of all possible worlds. It is analogous to the Leibnizian Palace of the Fates (*Theodicy*, secs. 414-417). But this leads into the metaphysics of modality.

5. The Roycean Metaphysics of Modality

Although it is well-known that one of Royce's students, Clarence Irving Lewis, went on to do extensive work in formalizing the metaphysics of modality, Royce himself does not get enough credit for his own work in the metaphysics of modality. For within both *The Conception of God* and *The World and the Individual* (especially the Supplementary Essay), there are extensive discussions of the metaphysics of modality.

Royce talks about counterfactuals (contrary-to-fact hypotheses) and alternative histories of our world (CG 193-200; SE 567-569, 573). When he is talking about the possible double of the hero in the story by Amadeus Hoffman (SE 574), Royce is talking about counterparts in other possible worlds. And Royce affirms that the Absolute, by virtue of its omniscience, is modally complete: it contains all possible worlds within its abstract thoughts (CG 198-199; SE 567-569, 580). Royce discusses these other possible worlds at length and explicitly refers to them as "possible worlds" (CG 212-216; SE 569).

Today it is well-known that the logical modes of possibility and necessity can be analyzed in terms of quantification over possible worlds. And C. I. Lewis was among the first to analyze necessity in terms of truth at every possible world.[11] Royce, of course, did not analyze these modes using quantification. Nevertheless, Royce is not just another thinker who vaguely talks about possible worlds. For Royce, through the Model, has provided *an explicit domain of modal quantification,* namely, the set of all possible thought-streams S. Hence the use of modern quantificational techniques to analyze modality is entirely consistent with Royce's metaphysics of modality. Clearly, this is consistent with modern analytic modal logic. Roycean modal logic quantifies over the set S of all possible thought-streams. For any proposition P, *it is possible that* P iff P is true at some thought-stream in S and *it is necessary that* P iff P is true at every thought-stream in S.

Within his metaphysics of modality, Royce distinguishes between those possible thought-streams that are *actualized* and those that remain *unactualized* (CG 200-203, 212-216; SE 573-576). Any unactualized thought-stream is a bare or unrealized possibility while any actualized thought-stream is a genuine or real possibility. Royce affirms that exactly one thought-stream is actual (CG 212, 214). Within the Model, this means that exactly one path $s(f, n)$ in the actual Kette (N, f) is an actual thought-stream. Since our physical world is actual, this thought-stream somehow corresponds to our physical world.

Royce defines the difference between the actual and the unactual in terms of the focus of the Absolute Self. Absolute Attention focuses on exactly one thought stream to the exclusion of all others (CG 200-203, 212-214; SE 569). Royce tells us that the self-focusing of the Absolute Attention is entirely free: "the attentive aspect of the Absolute Experience appears as itself possessed of absolute Freedom" (CG 202) and this is a "transcendent Freedom" (CG 203). This freedom is the expression of the Absolute's own self-definition as Absolute Love (CG 215-216). The actuality of this world instead of some other is "the actual Divine Love for *this* world" (CG 216, italics by Royce).

This loving focus of the Absolute Attention specifies the Absolute Will (CG 212; SE 573, 581n). Hence the difference between unactualized and actualized possibilities is the difference between unwilled and willed possibilities (SE 573-576). For any possible thought-stream s in any possible absolute self, s is an unactualized possibility if s is thought but not willed while s is an actualized possibility if s is both thought and willed. Within the Model, for any path $s(f, n)$ in any Kette (N, f), the path $s(f, n)$ is unactual if thought but not willed and actual if thought and willed. Of course, the unique actual thought-stream lies within the set of paths in the Kette of the actual absolute self.

6. Constructing the Physical World

On the one side, Royce argues that our world is ultimately an ideal thought-stream of the form $\langle n, f(n), f(f(n)), \ldots \rangle$. As such, it is a well-ordered discrete series of natural numbers. On the other side, Royce acknowledges that modern mathematical physics says that the world is accurately described by a system of differential equations (WI2 224-225). As such, it is a system of material particles moving along rigidly defined continuities. Of course, as an idealist, Royce affirms that the *physical world* is to the *ideal world* as surface structure to deep structure.[12] If that affirmation is to be anything but idle, then Royce must show how to construct the physical world from the ideal world. But how to carry out this construction? Royce is well-aware of the difficulties involved in answering that question (WI2 IV). And the fact that Royce's Model of the Absolute is already mathematical enables his idealism to answer this question in a scientifically serious way.

According to Royce's Model, to be an ideal world is to be a thought-stream in S. As a thought-stream, each ideal world has a deep *temporal* character. Time in any ideal world is a series of stages that is well-ordered like the natural numbers (WI2 137-139, 146-147). Ideal time is therefore both discrete and one-way infinite (it begins with an initial ideal event at the initial ideal moment corresponding to the initial natural number 0). As a thought-stream, each ideal world also has a deep *causal* character. Its causal character manifests itself on the physical surface as a system of invariant laws (WI2 188-191). Royce also says that the causal character of any thought-stream is purposive (WI2 192). The purposiveness of the ideal world manifests itself as irreversibility (WI2 101), which modern mathematical physics describes using the laws of thermodynamics (WI2 216-219).[13]

As the Absolute Will actualizes a thought-stream, it also actualizes the physical surface structure that emerges from that ideal deep structure. Absolute Insight sees the emergence of this physicality by constructing it (SE 573; WI2 52, 104, 133-151). For every thought-stream in S, there are many ways for Absolute Insight to constructively work up its numerical content into some basic physical world. Over any thought-stream, the Absolute decisively selects the construction of one basic physical world (SE 573). Any basic physical world is $\langle Q_1, .. Q_m, E_1, ... E_n \rangle$ where the Q_i are sets of basic physical objects and the E_i are basic physical properties and relations.[14] The Q_i and E_i must be derived from the numbers in the thought-stream $s(f, n)$ by mathematical construction. For example, given some thought stream $s(f, n)$, the Absolute may use its contents to construct a basic physical world $\langle Q, E_1, E_2, E_3 \rangle$ where Q is a set of space-time points, E_1 is a spatial neighbor relation among those points, E_2 is a temporal neighbor relation, and E_3 is a matter-field indicating the absence or presence of a particle at a point. These space-time points, their neighbor relations, and their field values, are derived from the numbers in the thought-stream $s(f, n)$ by the mathematical constructive activity of the Absolute.

To move from an ideal world to a physical world, Royce uses *recursive mathematical constructions*. The early lectures in WI2 explain these recursive constructions in detail. Physical time is constructed from ideal time. If physical time is discrete, then the construction is identity: the *n*-th physical moment is identical to the *n*-th ideal moment. However, if physical time is dense or continuous, then the construction must go beyond identity. Here Royce shows, using the recursive schema of interpolation, how a dense physical series of moments can be built up from a discrete ideal series (WI2 62-95).

Royce explicitly illustrates this schema (WI2 84-86). He starts with a pair of points $\langle a, b \rangle$; between them, he inserts a third point m, to make the triple $\langle a, m, b \rangle$. Within this triple, m can be interpreted as being *half-way* between a and b. He now repeats the operation of inserting points between pairs: he inserts m_1 between a and m, and m_2 between m and b, thus making $\langle a, m_1, m, m_2, b \rangle$. Royce says the result is "a series resembling a collection of points in order on one line" (WI2 85). He explicitly shows this in his text as:

$$a \;.\;.\;.\;.\; m_1 \;.\;.\;.\;.\; m \;.\;.\;.\;.\; m_2 \;.\;.\;.\;.\; b.$$

Within this illustration, the *quarter points of a physical line* segment now supervene on the ideal items. Thus 0 supervenes on a; 1/4 on m_1; 1/2 on m; 3/4 on m_2; and 1 on b. Each next repetition of the interpolation constructs an ideal series on which a more finely divided physical line segment supervenes. The next interpolation makes

$$a \;.\;.\; m_3 \;.\;.\; m_1 \;.\;.\; m_4 \;.\;.\; m \;.\;.\; m_5 \;.\;.\; m_2 \;.\;.\; m_6 \;.\;.\; b.$$

As a result of this interpolation, the points of a physical line segment that is divided into *eighths* now fall on these ideal items. Thus 0 falls on a; 1/8 on m_3; 1/4 on m_1; 3/8 on m_4; and so on. Each next interpolation doubles the refinement of the physical line; in the limit, the result is an ideal series of points on which a dense physical line supervenes. By analogous reasoning, Royce argues that *any* dense physical order can be constructed from some discrete ideal order. He mentions that orders for "distances, times, masses, temperatures, pressures" can be thus constructed (WI2 75).

Although this recursive construction is fine as far as it goes, it hardly goes far enough. The physical world is obviously far more complex than any collection of dense physical lines (that is, dense dimensions for physical quantities). For the physical world is a spatio-temporal-causal system involving apparently material things. Royce is well-aware that modern mathematical physics uses a system of differential equations, which are both continuous and non-purposive, to accurately describe the physical world (WI2 224-225). If Royce wants his metaphysics to be taken seriously in the light of modern mathematical physics, then he must show how an apparent but realistic system involving material things in spatial, temporal, and causal relations can be constructed from an ideal temporal-causal series. He must show how to use a discrete and purposive series of natural numbers to construct an apparently continuous and non-purposive physical world.

The construction is made problematic by the fact that, according to Royce's Model of the Absolute, every ideal temporal-causal series is a series of natural numbers. How can a realistic physical world be constructed from a series of natural numbers? Royce must show how to use a thought-stream $s(f, n)$ to construct worlds with multi-dimensional spaces containing material things regulated by causal laws. Royce does affirm that "the infinitely numerous properties of the numbers need some concrete representation" (SE 572). Unfortunately, Royce does not provide any such concrete representations – he does not provide any physical constructions.[15] It is necessary to provide them for him. Fortunately, this is not hard to do, given his discussion of the natural numbers in the *Essay*.

7. The Construction of Basic Physics

Against the critics who declare that the natural numbers are sterile and boring, Royce tells us that every natural number contains rich inner depths (SE 576-577). His discussion of the *perfect numbers* shows that Royce is interested in these depths (SE 577).[16] The internal meaning of any number can be unpacked through mathematical analysis.

One way to unpack the internal meaning of numbers is to look at how they are represented by strings of digits. Any natural number has a representation as a series of decimal digits (0 through 9). In the decimal system, every number is defined as a sum of powers of ten. Thus 205 is defined as the sum of 5 ones, 0 tens, and 2 hundreds (and 0 thousands, 0 ten-thousands, etc.). More formally, the number 205 is the series ⟨5 ones, 0 tens, 2 hundreds, 0 thousands, 0 ten-thousands, . . . ⟩. Or just ⟨5, 0, 2, 0, 0, . . . ⟩.

Although we represent numbers using decimal digits, computers represent them using binary digits (as *bit strings* made of 0s and 1s). Rather than using powers of ten, the binary system uses powers of two. The powers of two are one, two, four, eight, sixteen, thirty-two, and so on. Thus 205 is defined as the sum of 1 one, 0 twos, 1 four, 1 eight, 0 sixteens, 0 thirty-twos, 1 sixty-four, and 1 one-hundred-and-twenty-eight (and 0 two-hundred-and-fifty-sixes, etc.). More formally, the number 205 is the bit string ⟨1, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, . . . ⟩. Since every natural number is finite, it has a representation as a bit string in which there are only finitely many ones.[17] It can be represented by a bit string in which there is some *rightmost* 1 followed by infinitely many 0s marching off to the right.

Any series of bits is a one-dimensional one-way infinite discrete space. It can be made dense by the interpolative construction offered by Royce. However, the space in our physical world has many dimensions. It is therefore necessary to show how to use a bit string to construct a space that has many dimensions and that is infinite in all directions. This is easily done using an idea based on *Ulam's Spiral* (Stein et al., 1964).[18] The idea is to wind the bit string around itself so that it forms a coil or spiral. This is analogous to taking a linear measuring tape and winding it up around itself into a spiral. As the tape is wound around itself, the spiral fills a two-dimensional space.

For example, the decimal number 9227 is $2^0 + 2^1 + 2^3 + 2^{10} + 2^{13}$, which is equivalent to the bit string $\langle 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, \ldots \rangle$. The left part of Figure 2 shows how to wind this bit string around itself to make a spiral. The right part of Figure 2 shows the two-dimensional discrete space (a grid) filled by this bit spiral. Of course, Figure 2 can only show a few turns of the spiral so that it fills sixteen cells. The construction is easily extended to three or more dimensions. For our sample physical construction, every bit in the binary representation of any number in $s(f, n)$ is a point in physical space-time; the location of bits on the grid defines the spatial neighbor relation among bits; and bits from any two temporally adjacent natural numbers in $s(f, n)$ are temporal neighbors.

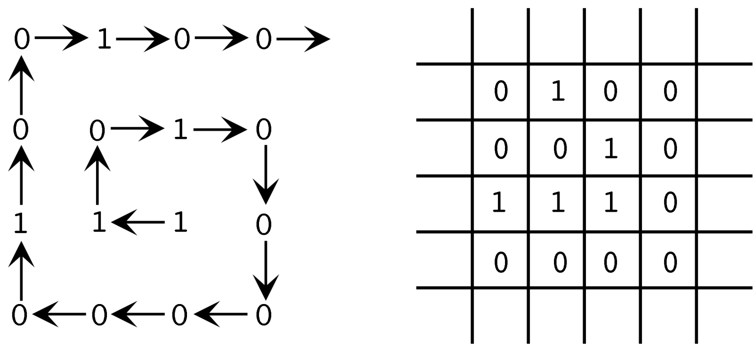

Figure 2. Winding a 1D string onto a 2D grid.

As any bit string gets wound around itself into a spiral, that winding distributes 0s and 1s to points in space. This distribution has a physical interpretation: a point is *occupied* by some bit of mass-energy if it is defined by a bit that is 1 and a point is *empty* if it is defined by a bit that is 0. The assignment of a bit of mass-energy to every point in space defines a *matter field*. Since the construction associates each point with a value 0 or 1, it defines a binary (Boolean) matter field. If the amount of mass-energy that is present at some point is 1, then a material *particle* is present at that point. Since each natural number, represented as a bit string, contains only finitely many 1s, it determines a matter field with only finitely may material particles. Hence this construction shows how a 2D discrete infinite space along with binary matter field supervenes on a natural number. Each of these structures is a complete *snapshot* of a very simple physical world at a time.

A temporal sequence of such snapshots is a (2+1)-dimensional discrete space-time with a binary matter field with finitely many particles.[19] Within recent analytic metaphysics, these structures are instances of *Democritean worlds* (Quine, 1969: 147-152). According to the absolute interpretation just constructed, a (2+1)-dimensional Democritean world supervenes on every possible ideal thought-stream. Of course, there are richer absolute interpretations involving more dimensions, dense dimensions, and matter fields with more that just two values. But those interpretations merely involve more technicalities; for our philosophical purposes, these simple Democritean worlds are enough.

For example, when the number 9227 is converted into a bit string and then wrapped around itself to make a spiral, the result is the leftmost 4 by 4 bit matrix shown in Figure 3. The number 9227 supports that bit matrix. Figure 3 also shows four other numbers, and the 4 by 4 bit matrices they support. The series of numbers in Figure 3 is one of the series in some thought-stream $s(f, n)$ in the Absolute. The horizontal arrows represent the iteration of f while the vertical arrows indicate that the number supports the matrix. Each 1 indicates some presence of matter (some particle) at its point in space while each 0 indicates the absence of any particle. Hence each matrix is a matter-field.

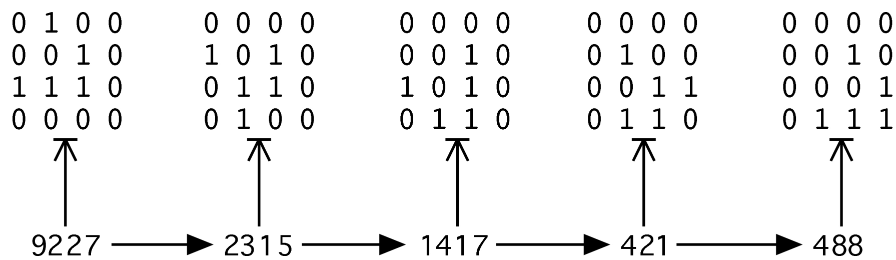

Figure 3. A series of matrices over a thought-stream.

8. The Construction of Material Things

This sample physical construction has defined a set of space-time points; a spatial neighbor relation on those points; a temporal neighbor relation; and a matter field. These basic physical objects and relations are sufficient for some very complex physics. Nevertheless, the movement from the ideal thought-stream $s(f, n)$ to the physical world remains incomplete: it does not yet include any construction of material things and their causal laws. To move from the ideal thought-streams in the Absolute Mind to material things regulated by causal laws, it is necessary to move through *computation*. It is worth noting that one of Royce's students, Norbert Wiener, links Royce to modern computing.[20]

Every ideal thought-stream is series of natural numbers; but any natural number can be represented as a bit string (a series of 0s and 1s). Any bit string can be represented as an ideal one-way infinite tape made of cells. Each cell contains either a 0 or a 1. These tapes can serve as the memories for simple computers like Turing Machines (Turing, 1936; Hopcroft, 1984).[21] Every *Turing computation* starts with some initial natural number n represented as a bit string written on a tape. And every Turing computation proceeds by iterating some function f from N to N. Each application of f to the previous tape yields the next tape. Hence every Turing computation is a path $s(f, n)$. From which it follows that every Turing computation exists in the set S of possible thought-streams in the Absolute. Some Ketten in the totality of possible absolute selves perform only Turing computations. These Ketten are ideal Turing Machines. And since the Absolute contains all possible $s(f, n)$, it contains thought-streams that compute non-Turing computable functions like the Busy Beaver function.[22] The Absolute is richer than mere Turing computing.

A Democritean world is *Turing-computable* iff every next state of space is derived from its previous state by some Turing-computation. The history of any Turing-computable Democritean world is the execution of an ideal Turing Machine. For every Turing-computable Democritean world, there is an ideal thought-stream in $s(f, n)$. Let T be the set of all *Turing-computable Democritean worlds*. The set T is a subset of S. The construction of physical worlds with material things and causal laws begins with the thought-streams in T. Turing-computability ensures that the thought-streams in T are not chaotic but are lawfully ordered. The set T contains every Turing-computable model of our best physical theories (including sophisticated models such as those of lattice quantum field theory). Hence Royceans can claim to have gone a long way to satisfying the demands of modern physics. The set T contains every computable substructure of our physical world. And it may contain the entire structure of our physical world.[23]

Every physical world in the set T supervenes on its underlying thought-stream as software supervenes on hardware. Every possible thought-stream in the Absolute is computationally foundational – it is the *hardware* relative to which everything else is *software*. Every physical world results from some absolute interpretation of some possible thought-stream.[24] Hence every physical world is a software world. It is a system of software objects. Software objects form layers. For any thought-stream that is interpreted as a Turing-computable Democritean world, the lowest software level is the level of the bit strings that supervene on the natural numbers. These bit strings are *first-level software objects*. They are the basic physical objects. Software objects can be stacked: lower-level software objects support higher-level software objects; conversely, higher-level software objects supervene on lower-level software objects. These higher-level software objects are derived physical objects. When the first-level bit strings are interpreted spatially using Ulam's Spiral, the result is a higher level of software objects. Hence each 2D spatial grid is a second-level software object (on a bit string on a natural number). Material things and their causal laws are found within these levels of software objects.

9. An Illustration of the Construction of Physics

Many Turing-computable Democritean worlds support rich stacks of higher-level software objects. It will be useful to focus on one genus of these worlds, namely, *cellular automata* (Chopard & Droz, 1998). A series of snapshots of the world is a cellular automaton iff there is one causal law that describes how the value of every spatial point changes over time. The causal law is a program that can contain many rules. A good way to illustrate a cellular automaton involves laying coins on a grid. Each square on the grid is a point (or *cell*) in space and each coin indicates the value of that cell (with heads equal to 1 and tails equal to 0). The distribution of coins defines a binary matter field.

The first illustration of a cellular automaton merely flips coins. To see it in action, start by placing one coin on each square of a tic-tac-toe board, randomly heads or tails up. Apply the following program to each coin: if the coin shows heads, flip it to show tails; if it shows tails, flip it to show heads. Once you've applied this rule to each coin, you've used the same program to change the value of the matter field at each point in space. You've updated the entire board, and thus you've generated the next snapshot of the physical world. The repeated updating of the board generates a temporal series of snapshots, and that temporal series is a cellular automaton. Each snapshot corresponds to a number, interpreted as a bit string that is spirally wrapped onto the tic-tac-toe board. Figure 4 shows a short thought-stream that supports one of these bit-flipping cellular automata. Figure 4 uses a black square for heads (for 1s) and a white square for tails (for 0s).[25]

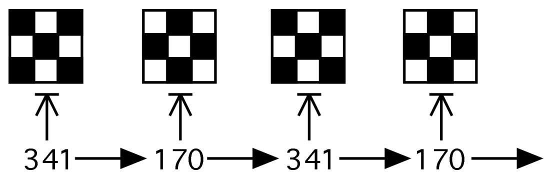

Figure 4. A short thought-stream supports a cellular automaton.

The second illustration of a cellular automaton is known as the *game of life* (Gardner, 1970). Since the causal law in the game of life involves some tedious book-keeping, it's best to watch it run on a home computer (many software packages for running it are freely available). Although the dynamics of the game of life look simple, they are literally infinitely rich, and the game of life has been the object of serious mathematical and philosophical study (e.g. McIntosh, 1991; Beer, 2004; Hovda, 2008).

To see a very simple example of the game of life, start by placing a coin tails up on each square of a 5-by-5 board. Pick a row of exactly three coins in the center of the board and turn them over so that they show heads. For each coin, depending on which side it shows, apply one of these rules: (1) if the coin shows heads, and it is surrounded by two or three neighbors that show heads, then it mark it as stable; otherwise, mark it as unstable; (2) if the coin shows tails, and it is surrounded by exactly three heads, then mark it as unstable; otherwise, mark it as stable. After you've marked each coin as stable or unstable, flip every unstable coin, and leave every stable coin as it was. If you repeatedly apply this program to the board with three heads in a row (and all others tails), you will notice that the orientation of the row of three heads alternates, as shown in Figure 5. Figure 5 uses a black square for heads (1s) and a white square for tails (0s).[26]

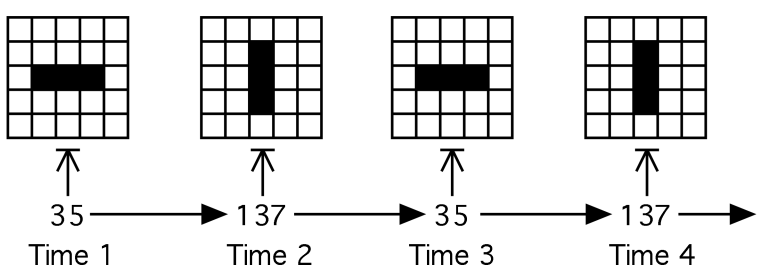

Figure 5. A thought-stream supports a trivial game of life.

For every possible game of life, there is an ideal thought-stream in the Absolute that supports that game. And games of life support rich stacks of software objects. Dennett (1991) says there are three main software levels in the game of life. The first level is just the bit strings that supervene on the numbers in the thought-stream. The second level is the *physical level*. The objects on this level are second-level software objects. At this level, the ontology consists of the space-time points (the "cells") of the life grid. The language of this level talks about points and their matter field values (0 or 1). It talks about spatial and temporal relations. It talks about the events that cause a point to change its state according to the basic causal law of the game of life. At this level, prediction concerns only the use of the basic life law to predict the states of points at the next time.

The third level is the *design level*. The objects on this level are third-level software objects; they are *patterns of activity* in space. Figure 6 shows the pattern known as the *glider*. The distributions or patterns of 1s in the matrices in Figure 3 correspond to the black squares (the active points) in Figure 6. This shows how a number in an ideal thought-stream supports an energetic pattern in space. Dennett says that the design level has its own ontology (1991: 39). It supports patterns that *persist*, *move* and *causally interact*. The glider has a direction and velocity. It is possible to write equations of motion for the glider that are *independent* of the equations for the changes of the matter field. These equations of motion are an example of a Roycean causal law (WI2 188-191).

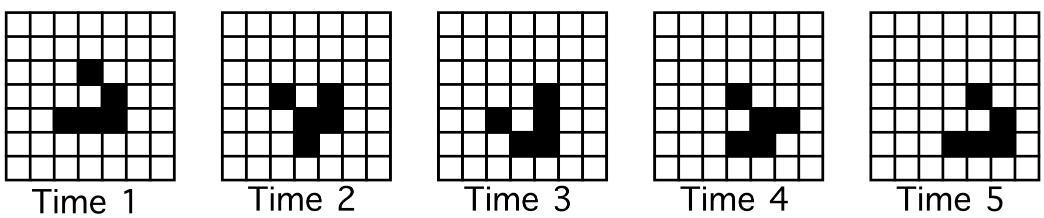

Figure 6. The apparent motion of a glider.

Things like gliders are on the lowest sublevel of the design level. The design level supports a part-whole relation. Wholes are higher-level patterns that supervene on networks of lower-level parts. Dennett says "one can set oneself the task of designing some interesting supersystem out of the 'parts' that the design level makes available." (1991: 40) According to Dennett (1991: 40), the most impressive result of this design activity is the construction of a universal Turing machine (a UTM). There is a proof that it is possible to construct UTMs on the life grid (Poundstone, 1985: 197-213). And Rendell has recently shown how to carry out the construction (2002).

At the very top of the hierarchy of sublevels within the design level, there is the level of the UTM itself. At this level, computational language is used to describe what is happening on the life grid. Dennett says "As a first step one can shift from an ontology of gliders and eaters to an ontology of symbols and machine states, and, adopting this higher design stance toward the configuration, predict its future *as* a Turing machine." (1991: 41) UTMs support many levels of their own. It is possible to move from talking about tape squares to talking about registers, arrays, linked lists, and other software objects. Given UTMs, it is possible to construct software objects that model complex psychologies.

The fourth level is the *intentional level*. Objects on this level are fourth-level software objects. Dennett imagines two UTMs interacting with one another in some game of life. Since UTMs can play chess, these two UTMs can play chess with one another. The result is that a novel taxonomy of objects emerges from the activity of the UTM. These objects include things like chess squares and chess pieces. Thus "one can shift to an ontology of chess-board positions, possible chess moves, and the grounds for evaluating them; then, adopting the intentional stance toward the configuration, one can predict its future *as* a chess player." (1991: 41) These chess players are themselves intentional – they are minds. They have psychologies. At this level, there are finite selves interacting in a small society. Any mind (or community of minds) that can be realized by some UTM exists as a fourth-level software object over some thought-stream in the Absolute. The construction of these UTMs shows exactly how finite selves can be encoded in the Absolute Self.

Although human selves are hardly as simple as the chess-playing UTMs described by Dennett, there are similarities. Our brains are at least finitely powerful computers. As our senses are bombarded with phenomenal stimuli, our brains seek to find patterns in them. We thus *project* into the phenomenal world those patterns that best serve our pragmatic interests. For instance, we project systems of differential equations into the phenomenal world (WI2 224-225). However, those equations are excessively precise and so cannot be taken literally. For example, the *Lotka-Volterra differential equations* describe the mutual fluctuations in a population of predators and their prey. Although those equations involve continuous variables, animal populations are in fact merely discrete sets. The equations are excessively precise. The *Navier-Stokes differential equations* for fluid motion likewise falsify that fluidity by treating a discrete population of moving particles *as if* it were a continuous substance. Advocates of *digital mechanics* support Royce by arguing that the continuity is a false projection into an ultimately discrete reality.[27]

10. Roycean Philosophy of Nature

Our physical world is far more complex than any game of life (and almost certainly far more complex than any cellular automaton). However, our world is similar to a cellular automaton insofar as it consists of a stack of software levels. Each level is a network of interacting software objects. But what are these software objects? I will work through four theses, based on Royce's texts, for the existence of software objects. The series converges to a fifth thesis that shows how finite selves exist in our physical world.

Thesis 1 says that for every human body H, there exists a *human mind* H* such that H is the sign of H* (N1; WI2 III). Royce says that "the very bodies of our human fellows" express themselves in "gestures, words, deeds" which "indicate the inner life of the social fellow-being who thus expresses himself" (N2 581). Every human mind has certain psychological features (e.g. consciousness, rationality, ethical engagement). Every human mind runs at the human time-rate. Sets of human minds form human social networks. The minds in those networks interact via linguistic signals. Human networks are organized according to certain structural principles (they have certain social topologies).

Although Thesis 1 is true, it is not the whole story (WI2 III). The experience of human social life leads to the conception of non-human nature (N1 470, 471, 472; WI2 IV.II; WI2 180). Non-human nature is continuous with human nature (N1 584, 587; WI2 V.I) and can only be understood through its continuity with human nature (N2 587; WI2 V.III). Royce writes that "the higher animals manifest their inner experience, apparently similar to ours, by expressive activities which resemble ours" (N2 590). And the continuity of inner life must be extended at least to every organism that has a nervous system (N2 591-592). Hence Thesis 1 must be generalized to make Thesis 2.

Thesis 2 says that for every organism O, there exists an *organic mind* O* such that O is the sign of O*. Royce writes that "All life, everywhere, in so far as it is life, has conscious meaning, and accomplishes a rational end" (WI2 240; Price, 1999). Every organic mind is similar to a human mind (it is conscious, rational, ethically engaged). Every organic mind runs at a time-rate defined by its species. Sets of organic minds form organic networks. The minds in organic networks interact by exchanging linguistic signals. Organic networks are organized on principles similar to those of human networks.

And yet Thesis 2 is still not the whole story. For if it were, then inorganic nature would be mindless and dead. However, there is no dead nature. For Royce says that "where we see inorganic nature seemingly dead, there is, in fact, conscious life" (WI2 240) and he says that there is no evidence for "the existence of dead material substance anywhere" (WI2 240; N2 586). On the contrary, on the hypothesis of idealism, "a material region of the inorganic world would be to us the phenomenal sign of the presence of at least one fellow-creature who took, perhaps, a billion years to complete a moment of his consciousness" (WI2 228). Thus nature is "the sign of the presence of other finite consciousness than our own" (WI2 228). Hence Thesis 2 must be generalized.

For every natural material thing M, there exists a *substantial mind* M* such that M is the sign of M* (N2 585, 586; WI2 213). Inorganic nebulae are minds (WI2 227); hydrogen atoms may be minds (WI2 230). Every substantial mind has a psychology like human psychology (N1 470, 471; WI2 228). Every substantial mind is conscious (WI2 225-226). Every substantial mind runs at a time-rate defined by its natural kind (N2 595-596; WI2 225-228). A mind may take "a millionth of a second . . . a minute, an hour, a year, a century, or a world-cycle" to complete one mental operation (N2 598). Sets of substantial minds form substantial networks. The minds in substantial networks interact by exchanging linguistic signals (N2 588). These languages are mainly unintelligible to us (WI2 230-231, 236-237). Yet the structural principles of substantial networks are similar to those of human networks (N1 472, 473; N2 584).

Of course, it may be objected that Thesis 3 is too general. It is not the case that every material thing is a mind (WI2 233). For instance, Royce says "I do not suppose that any individual thing, say this house, or yonder table, is a conscious being" (WI2 233). To qualify Thesis 3, it is necessary to look more closely at the presence of regularity and irregularity in natural processes. As natural processes tend toward pure regularity, they also tend toward unconsciousness (N2 592-594). Mental life reveals itself, not by the pure regularity of habit, but by the "alteration of old habits to meet new conditions" (N2 593). And, according to Royce, when natural processes are viewed on the right time-scales, they in fact exhibit this adaptive alteration of habit (N2 594-596).

For convenience, let us simply say that a process is *adaptive* iff, on the appropriate time-scale, it displays an alteration of old habits to meet new conditions. Thesis 3 can now be qualified to say that every adaptive process is the sign of a mind. This new version of Thesis 3 is illustrated by animals. Individual animals are signs of animal minds. But individual animals of the same species are signs of a single species-mind (WI2 232, 241). On large biological time-scales, this species-mind shows itself as adaptive. Note that Royce allows minds to be parts of minds (WI2 238, 303-304); hence there is nothing wrong with saying that an individual animal mind is part of the species-mind.

There is no reason to restrict mental life to processes that are similar in size or duration to human bodies and lives. Every region of space may be animated by a mind (WI2 232). The smallest and fastest processes (e.g. at the atomic level) may be animated by minds (WI2 213, 230) while the largest and slowest processes (e.g. at the geological or astronomical levels) may be animated (WI2 227-228). The turns of the earth, the orbits of the planets and the dynamics of the solar system may be signs of minds (N2 596). The larger and slower minds may be more rational than human minds (WI2 231). By analogy with animal species, even corporations and nations may be minds.[28]

All these considerations motivate Thesis 4: for every adaptive process K, there exists a *natural mind* K* such that the activity in K is the sign of K*. Every natural mind is has a psychology that is more or less similar to human psychology (e.g. it is conscious, rational, and morally involved with others). Every natural mind runs at its own time-rate. Sets of natural minds form natural networks. The minds in natural networks interact via linguistic signals. Natural networks are organized on principles similar to those of human networks. For example, many human networks are *scale-free* or *small-world* networks (Barabasi & Bonabeau, 2003). These sorts of networks are found throughout nature.

Although Royce indicates that pure regularity is the mark of unconsciousness, it may be objected that he went too far. It seems more consistent with contemporary computational analyses of mentality to say that any algorithmically defined process (any computation) displays some non-zero degree of mentality (Doyle, 1991). Purely regular processes have minimal (but non-zero) degrees of mentality. And while Royce focuses on consciousness, it may be objected that more modern psychology has shown that not all mental processes are conscious processes. For consistency with current science, the psycho-social concepts in Thesis 4 should be translated into computational concepts. This translation does no damage to the essential content of Royce's idealism. On the contrary, it merely develops that content in a more modern and scientifically comprehensible way.

The computational translation of Thesis 4 changes it in four ways. The first way is that adaptive processes are replaced by algorithmically defined processes (by computations). The second way is that minds are replaced by software objects (which need not be conscious, rational, or morally involved). The third way is that languages are replaced by semiotic systems (which need not be much like human languages). The fourth way is that networks of software objects need not be much like human social networks.

When Thesis 4 is translated, the result is Thesis 5: for every algorithmically defined process W, there exists a *natural software object* W* such that the activity in W is the sign of W*. These processes may be as small as points or as large as space-time itself. Every natural software object is an information processor that has more or less similarity to natural minds. Every natural software object runs at its own time-rate (its own clock-speed). Sets of natural software objects form natural networks. The software objects in natural networks interact by exchanging signs in semiotic systems that are more or less like human languages. The natural networks are more or less similar to human social networks.

All the work done so far suggests a fourth point of contact between Royce and current analytic metaphysics. One school of thought in current analytic metaphysics is known as *structuralism*. Structuralists say that reality is a system of pure relations (Dipert, 1997). It has been suggested that Royce is a structuralist (Crouch, 2011). One way to develop this suggestion looks like this: societies are purely relational structures of minds; minds are purely relational structures of thoughts; and thoughts are positions in purely relational semiotic systems.[29] Of course, if the Model of the Essay is taken seriously, then all these structures supervene on numerical structures (e.g. Ketten of the form (N, *f*)). But the number system itself is purely relational (Resnik, 1997; Shapiro, 1997). Here the contact between Royce and current analytic metaphysics is deep and fascinating.

11. Conclusion

At the very least, formal Roycean metaphysics involves the study of the structures that supervene on the iterations of the self-maps of the natural numbers. But there are three main ways to further develop formal Roycean metaphysics. The first way includes the further developments of the Model within Royce's own metaphysics. In the last lecture of WI2 (Lecture X), Royce uses the Model to illustrate his theory of immortality. He also refers to the Model in *The Conception of Immortality* (1900: 82-90). The second way to further develop the Model involves the extension of the Model to greater ordinals. The set of natural numbers is the least transfinite ordinal; but modern mathematics defines many greater ordinals (Hamilton, 1982: ch. 6). Formal Roycean metaphysics includes the study of the structures that supervene on the iterations of the self-maps of greater ordinals. These structures are currently studied in the theory of transfinite computation (e.g. Hamkins & Lewis, 2000; Koepe & Siders, 2008). The third way to further develop the Model involves deeper study of the ways it makes contact with recent analytic metaphysics and natural science. There is much work that can be done in each of these three ways. Formal Roycean metaphysics offers spectacular opportunities for deep mathematical, metaphysical, and scientific research. It is a paradise waiting to be explored.

[1]Royce (1897) contains the texts of the great debate. It includes papers by Royce, Joseph Le Conte, G. H. Howison, and Sidney Mezes. I will use the following abbreviations to refer to frequently cited works by Royce: *N1* stands for Royce (1895A); *N2* stands for Royce (1895B); *CG* stands for Royce (1897); *SE* stands for the Supplementary Essay (Royce, 1899); and *WI2* stands for Royce (1901).

[2]From the start to the finish of Volume 2 of *The World and the Individual*, Royce refers back to Supplementary Essay of Volume 1. See WI2 18, 67, 75, 76, 83, 96-98, 105-106, 146-147, 297-298, 303, 306, 446-451.

[3]The reception of the metaphysical Model in the Essay was mixed: Peirce (1902), Leighton (1904) and Swenson (1916) are mainly negative; Loewenberg (1916) is positive; Haldar (1918) is highly positive; Monsman (1940) is critical but generally positive. The Essay is barely mentioned in Jarvis (1975: 67-72). None of these authors develop Royce's metaphysical Model any further. Royce's map of England is often noted in discussions of the history of the concept of infinity (e.g. Rucker, 1995: 38-41; Moore, 2001: 101-103). Yet as Price (1999) notes, the Essay, and the metaphysics it inspires in the Second Volume of *The World and the Individual*, is buried in silence.

[4]The iteration of f can be used to define a series of functions. Formally, for any n, $f_0(n) = f(n)$; for any k and for any n, $f_{k+1}(n)$ is $f(f_k(n))$. The *range* of any function is the set of outputs of the function. Royce observes that for any k, the range of each function f_{k+1} is a proper subset of the range of the previous function f_k (SE 523-525). For example, the range of f_0 is {0, 1, 2, 3, . . . }. The range of f_1 is {0, 2, 4, 6, . . .}. The range of f_2 is {0, 4, 8, 12, . . . }. The range of f_3 is {0, 8, 16, 24, . . .}. So the range of f_{k+1} is always a proper part of the range of f_k. These ranges form a nested series. The function f_k corresponds to the k-th nested map of England within England.

[5]On the one hand, a set is *Dedekind infinite* iff there is some one-to-one correspondence between the set and one of its proper subsets (SE 521). On the other hand, a set is *infinite* iff it can be put into a one-to-one correspondence with some infinite cardinal number. This is a subtle technical distinction. Since the set of natural numbers N is both infinite and Dedekind infinite, so long as Royce focuses on Ketten involving self-maps of N, this subtle distinction does not matter. But what about Ketten involving other Dedekind infinite sets? For modern mathematics, based on Zermelo-Fraenkel-Choice set theory, it is a theorem that a set is Dedekind infinite if and only if it is infinite (Hamilton, 1982: 168-169). This theorem requires the Axiom of Choice. The Axiom of Choice is equivalent to the thesis that every set can be well-ordered (Hamilton, 1982: 172). Royce seems to affirm this thesis (WI2 89). If Royce really does affirm this well-ordering thesis, then it is sufficient to say that Royce is interested in Ketten in which the set is infinite.

[6]For Royce's purposes, it does no harm to allow that a Kette is any (N, f) in which f is a one-to-one map from N onto *any* subset of itself (whether proper or improper). Since

Royce wants to avoid the inexactness produced by the fusion of many items into one item (SE 521), he merely needs to avoid Ketten in which f is many-to-one.

[7]It was quickly observed that any interpretation of these Ketten as selves is fraught with difficulties (Peirce, 8.125; Leighton, 1904; Santayana, 1920: 135-137).

[8]It may be argued that each of these infinite forms of selfhood is the complete meaning of a finite human self. However, that meaning is not a finite human self. It is what Royce refers to as an *Ethical Individual* (WI2 445-452). The formalization of the Model developed here is consistent with the existence of these Ethical Individuals.

[9]On the interpretation advanced here, the thesis that the Absolute "*defines itself* as self-representative" (SE 546, my italics) entails the thesis the Absolute Self is "a *self-selected* case of its type" (SE 566, my italics). Thus exactly one Kette (N, f) is self-selective within the set of all such Ketten. Unfortunately, Royce is not clear about this.

[10]Prior to developing the Model, Royce says that the total system of possibilities forms a group (CG 208-211). The structure (F, •, I) is a semigroup. The semigroup operator • is functional composition; the identity element I is $I(n) = n$ for all n in N. The structure (F, •, I) is a semigroup (and not a group) because not every f in F has an inverse. For the theories of semigroups and groups, see Stoll (1979: secs. 8.2 and 8.3).

[11]When he is analyzing the necessary truth of the laws of arithmetic, C. I. Lewis says that they "would be true in any possible world" (1923A: 172). Lewis goes on to define possible worlds as sets of facts (1923B; see Sedlar, 2009). Thus Lewis provides a bridge from Royce to more modern analytic theories of modality.

[12]Royce refers to the physical world as the *World of Description*. He typically contrasts the World of Description with the *World of Appreciation* (e.g. WI2 26, 46, 155-156, etc.). On the interpretation developed here, the World of Appreciation is encoded in the Absolute Will via the construction that gives rise to the World of Description.

[13]Any thought-stream is $\langle n, f(n), f(f(n)), \ldots \rangle$. The repeated application of the ideal law f to its own results over and over again generates the causal laws of the sequence. However, the iteration of f is never exhausted by any fixed system of causal laws (WI2 191-192). Thus the iteration of f is *purposive*. Out of its own history, the iteration of f perpetually generates *novel* patterns of causality involving new objects. As time goes by, layers of ever more complex physical patterning *emerge* from earlier and simpler layers. For emergence, see Bedau (1997). Any thought-stream thus supports an *arrow of evolutionary complexity* (WI2 V). Any thought-stream expresses a single plan or purpose (WI2 146-147). Now Royce says that "Whatever expresses a single purpose has, as the expression of that purpose, an irreversible succession" (WI2 101). This irreversibility manifests itself as the thermodynamic *arrow of time* (WI2 216-219). The purposive flow of absolute thought manifests itself in the evolution of further physicality (WI2 219).

[14]Any basic physical world serves as the basis for an objective system of classification (WI2 52). This system provides the objective standard of truth against which our subjective statements must be compared for truth or falsity. From the point of view of modern analytic metaphysics, this objective system of classifications is a formal semantic structure that supervenes on the basic physical world. This formal semantic structure is the image of an objective reference function. It therefore underwrites the truth-conditions that express what it means for a proposition P to be true at a thought-stream in S. The

structure of our actual world is defined by the Absolute Will (WI2 137-139, 148). This structure is objective: "God distinguishes what it pleases him to distinguish. The logical as well as the moral problem is, Does my will accord with God's will?" (WI2 52).

[15]Royce talks about space and geometry (e.g. WI2 66-67, 78, 90-95, 100). Yet he offers no clues on how a realistic space supervenes on a discrete temporal-causal series.

[16]A natural number is *perfect* iff it is the sum of those factors of itself which are less than itself. For instance, the factors of 6 less than 6 are 1, 2, and 3; but 6 = 1 + 2 + 3; so 6 is a perfect number. The next perfect number is 28.

[17]Formally, any natural number n is represented by $\langle b_0, b_1, b_2, \ldots \rangle$ iff

$$n = \sum_{i \in N} b_i 2^i \ .$$

[18]Ulam's original spiral starts with 1 and runs counterclockwise. While Ulam was looking for patterns of prime numbers, we are merely mapping a 1D series onto a 2D grid. For our purposes, the spiral can start with 0 and run clockwise or counterclockwise.

[19]An $(m+n)$-dimensional space-time has m spatial dimensions and n temporal dimensions.

[20]Norbert Wiener studied with Royce at Harvard from 1911 to 1913 (Wiener, 1948: 1). An initial study of the Royce-Wiener connection can be found in Peters (2010: ch. 2). Wiener went on to work with Bertrand Russell, David Hilbert, and John von Neumann.

[21]Since every tape for a Turing Machine (TM) has only finitely many 1s, it is equivalent to a natural number. And any action of the controller on the tape changes at most one bit on the tape. It therefore produces another natural number. So the action of any TM realizes a map from N to N. Specifically, every TM is a Kette (N, f) such that f is an arithmetically recursive function from N to N. Note that a Turing Machine with a one-way tape is equivalent to one with a two-way infinite tape (Boolos, 1989: ch. 3).

[22]The thought-streams in the Absolute contain more than just Turing computations. Giunti (1997) has defined computers that operate on standard tapes but that are more powerful than Turing Machines (they can jump directly to tape squares indexed with Rado numbers, and thus compute the Turing-uncomputable Busy Beaver function; see Boolos, 1989, ch. 4). The set of *Giunti computations* includes but exceeds the set of Turing computations. And since every Giunti computation is a path $s(f, n)$, every Giunti computation is a thought-stream in the Absolute. But the set of functions from N to N is even more general. More generally, every thought-stream $s(f, n)$ is a *Royce computation*.

[23]Three lines of reasoning support the thesis that our universe is a computation. The first line is theoretical. Many physicists have argued that the physics of our universe is ultimately computational (Fredkin, Landauer, & Toffoli, 1982; Deutsch, 1985; Fredkin, 1991; Zeilinger, 1999; Fredkin, 2003). The second line comes from studies of the informational capacity of our universe. Bekenstein & Schiffer (1990) argue that any finite quantity of matter can encode only finitely many bits of information. Lloyd (2002) calculates that our universe so far has run only 10^{120} operations involving 10^{120} bits. These are finite numbers. If our universe is only finitely complex, then it is computational. The third line is statistical. Zenil & Delahaye (2010) developed statistical methods for testing the hypothesis that processes within our universe are algorithmically generated. Examining various physical data sets, they discovered correlations that support the hypothesis. If all the processes in our universe are algorithmic, then our universe is computational.

[24]Some writers think that every possible software process supervenes on every hardware process (e.g. Putnam, 1988: 120-125; Searle, 1990: 25-27). However, more careful logical work refutes that promiscuity (Copeland, 1996; Chalmers, 1996; Scheutz, 1999). It is not the case that every higher-level process supervenes on every lower-level process.
[25]Using a spiral to wrap a bit string onto the grid, the figure that looks like an X is $2^0 + 2^2 + 2^4 + 2^6 + 2^8$, which is 341; the diamond is $2^1 + 2^3 + 2^5 + 2^7$, which is 170.
[26]Using a spiral to wrap a bit string onto the grid, the horizontal bar is $2^0 + 2^1 + 2^5$, which is 35; the vertical bar is $2^0 + 2^3 + 2^7$, which is 137.
[27]Advocates of digital mechanics say that our physical world is ultimately a digital system; it is a computation (which may be either finite or infinite). Advocates of digital mechanics include writers like Fredkin (1991, 2003), Toffoli (1984, 1990), Schmidhuber (1997), Zeilinger (1999), Wolfram (2002), and many others.
[28]An analogy (not developed by Royce) clarifies the relation of individual animal minds to the species-mind: just as old cells in our bodies are constantly replaced by genetically type-identical new ones, so the old individuals in an animal species are constantly replaced by type-identical new ones. Organic wholes persist by replacement of type-identical parts. Continuing this analogy, super-organisms (e.g. insect colonies), corporations, and nations are organic wholes – they are software objects.
[29]Structuralist approaches to societies and languages are already well-known. One widely accepted thesis in analytic philosophy of mind says that *minds are software objects*. This thesis is known as *functionalism* (Block, 1980). Shapiro (1997: 106-108) shows that functionalism about minds is a kind of psychological structuralism.

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