Relations

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1. The Genetic Code

The machinery of the cell translates strands of RNA into proteins. A strand of RNA is a sequence of bases. There are four types of bases: Uracil, Cytosine, Adenosine, and Guanine. We can refer to each type by its first letter. Thus the set of base types is:

$$B = \{U, C, A, G\}.$$

The translation machinery of the cell works with triples of types of RNA bases. Any triple of RNA bases is known as a *codon*. The set of codons is

$$\text{Codons} = B^3 = B \times B \times B.$$

The translation machinery associates each triple of RNA base types with either a type of amino acid or with an instruction like Stop. There are 21 types of amino acids. It's standard to refer to amino acid types using three letter abbreviations. For example, *Ile* is *isoleucine*. We give the abbreviations but not the full names. If we include the Stop instruction in the set of amino acid types, we get

Aminos = { Ala, Arg, Asn, Asp, Cys, Gln, Glu, Gly, His, Ile,
Leu, Lys, Met, Phe, Pro, Ser, Thr, Trp, Tyr, Val, Stop}.

The association from codons to amino acid types (including the Stop instruction) is known as the *genetic code*. It is a function of the form

$$f: \text{Codons} \rightarrow \text{Aminos}.$$

The genetic code is displayed in Table 1. Each row is the first base in a codon; each column is the second base; each row in a cell is for the third base.

	U		C		A		G	
U	U	Phe	U	Ser	U	Tyr	U	Cys
	C	Phe	C	Ser	C	Tyr	C	Cys
	A	Leu	A	Ser	A	Stop	A	Stop
	G	Leu	G	Ser	G	Stop	G	Trp
C	U	Leu	U	Pro	U	His	U	Arg
	C	Leu	C	Pro	C	His	C	Arg
	A	Leu	A	Pro	A	Gln	A	Arg
	G	Leu	G	Pro	G	Gln	G	Arg
A	U	Ile	U	Thr	U	Asn	U	Ser
	C	Ile	C	Thr	C	Asn	C	Ser
	A	Ile	A	Thr	A	Lys	A	Arg
	G	Met	G	Thr	G	Lys	G	Arg
G	U	Val	U	Ala	U	Asp	U	Gly
	C	Val	C	Ala	C	Asp	C	Gly
	A	Val	A	Ala	A	Glu	A	Gly
	G	Val	G	Ala	G	Glu	G	Gly

Table 1. The genetic code.

2. Vagueness and Fuzzy Sets

Given some universe of individuals U, we can define sets over that collection using *characteristic functions*. A characteristic function is a function from the universe U to the numbers in {0, 1}. Or, equivalently, from U to the truth-values in {F, T}. Suppose f is a function from U to {0, 1}. The function f defines a set over U. For any x in the universe U, we define the membership of x in f like this:

x is a member of f if $f(x) = 1$;

x is not a member of f if $f(x) = 0$.

A characteristic function expresses the idea that an object is either a member of a set or else it is not a member of a set. It expresses the idea that a set is formed by some definite yes or no decisions. There's nothing new in this way of thinking about sets.

It may happen that we want to define *degrees of membership* in a set. A predicate like being a prime number is clear enough: either x is a prime number, or it is not. But in the world of ordinary language, predicates are often not so clear. Predicates are often *vague*. The most famous example of a vague predicate is *baldness*. It's clear that old Abe is bald. And Bob has a head covered with hair. Bob is definitely not bald. But what about

Charles? Most of his hair is gone. But he still has some. Is he bald or not? It's probably not worth fighting about it. It's almost certainly better to say that there are *degrees of baldness*. Let B measure the degree to which x is bald. Then:

B(Abe) = 1 since Abe is bald;
B(Bob) = 0 since Bob is not bald;
B(Charles) = 1/2 since he's sort of bald.

Fuzzy Sets. Given any universe of objects U, a fuzzy set over U is a function from U to some set of degrees. At one extreme, the degrees {0, 1} define just ordinary sets. At another extreme, we can use real numbers between 0 and 1 as degrees. The set of real numbers between 0 and 1 inclusive is denoted [0, 1]. If we're using these as degrees of membership, then a fuzzy set over U is a function from U to [0, 1].

Fuzzy sets can be used to analyze vague predicates (like *bald* or *tall*). And they can also be used to analyze vague relations. For example, van Inwagen (1995: 221 – 227) says that the parthood relation is vague, and he uses fuzzy sets to analyze that vagueness.

3. Quine's Democritean Worlds

Quine defines a set of numerical structures that have been called *Democritean worlds* (Quine, 1969: 147 - 152). Cellular automata like the game of life (and lattice gasses and lattice Boltzmann structures) are Democritean worlds with finitely extended and divided space-time. A Democritean world is a "cosmic distribution of binary choices (occupied vs. empty) over the points of space-time" (Quine, 1969: 155). Occupied means occupied by matter or energy. In the game of life, occupied is the cell-value 1 while empty is 0. We cite Quine's development of Democritean worlds at some length:

> let us accept for a while an old-fashioned physics according to which, as Democritus held, all atoms are homogeneous in substance and differ only in size, shape, position, and motion. Let us suppose further that space is Euclidean. . . . there remain for each point in space just two possible states: the point may lie within some particle or it may be empty. Each distribution of these states over all the points of space may be seen . . . as a possible momentary world state. . . . The virtue of taking a possible world state as an exhaustive assignment of 'occupied' or 'empty,' 'yes' or 'no,' to points of space, is that the assignment . . . can simply be identified with the aggregate of the occupied points themselves. Each portion of space, big or little, compact or scattered, may thus be accounted a possible world state. Realization of that world state would consist in there being matter at each of those points of space and none elsewhere. . . . we can by-pass the points by adopting a system of coordinates and speaking of triples of real numbers. Our ontology then requires only portions of matter, as individuals, and the usual superstructure of classes of individuals, classes of classes, and so on. . . . a possible world state becomes simply any class of triples of real numbers. To any such class we equate what, intuitively, would be called the possible world state

> that has matter at just the positions given by number triples in the class. . . . This explication of possible worlds is predicated on the view that every possible world has homogeneous matter, Euclidean space, and a time dimension independent of frame of reference. these traits, being then traits of all possible worlds, rate as necessary. This view is debatable, since the real world is believed to lack all three traits. One thing good about this version of possible worlds, nevertheless, is that it stays within a clear extensional ontology. I expect that while still staying within these terms we could complicate it to suit current physics. (Quine, 1969: 147 - 152)

Quine starts out by saying that a momentary state of a Democritean world is a pair (P, f) where P is a set of spatial points and f is a function from P to {0, 1}. In physics, a field is a function that associates every point in space with an object. So the function f is a *matter field*. For any point p in space, Quine says

$f(p) = 0$ if p does not lie within any material particle (it is empty);

$f(p) = 1$ if p lies within some material particle (it is occupied).

The set of all possible momentary world states is the set of all functions from P to {0, 1}. If we use S to denote the set of momentary world states, then

$$S = \{ f \mid f: P \rightarrow \{0, 1\}\}.$$

Presumably, a Democritean world is temporally extended. It is a time-ordered series of momentary states. It is therefore a quadruple of the form (P, T, S, W) where P is a set of points of space; T is a set of moments of time; S is the set of possible momentary world states; and W is a function that associates each moment of time with a momentary world state. But Quine wants to take this further.

Each matter field is a characteristic function over the points of space. It selects some points and rejects others. Every matter field corresponds to a region of space. A region of space is a subset of P. So every matter field corresponds to a subset of P. On the basis of this correspondence, Quine redefines momentary world states as regions of space. The set of all possible momentary world states is now the power set of P. He says

$$S = \text{pow } P.$$

Quine further says that points can be replaced by triples of real numbers. Assuming that R refers to the set of real numbers, the set of points is now redefined as

$$P = R^3.$$

Following this redefinition of spatial points, we can replace each moment of time by a real number. Thus

$T = R$.

Given these replacements, a Democritean world is a quadruple of the form (R^3, R, pow R^3, W). If we follow the Quinean replacements, then we have redefined a Democritean world to be a purely set-theoretic structure over the real numbers.

Of course, you might raise some objections to these Quinean replacements. The most obvious objection is that a matter field is *not the same kind of thing* as a region of space. It is true that there is a 1-1 correspondence between the occupied portions of matter fields and the regions of space. But that hardly entails that they are identical. The same sort of objection can be raised against the replacement of points with triples of real numbers. Do you think these objections are the same in both cases? What are your replies?

References

Quine, W. V. (1969) *Ontological Relativity and Other Essays*. New York: Columbia University Press.

van Inwagen, P. (1995) *Material Beings*. Ithaca, NY: Cornell University Press.